Geodynamics
Lecture 3
Forces and stresses

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Goals of this lecture

- Define forces and stresses
- Discuss how stresses can be represented graphically and mathematically
- Look at various useful stress relationships and their geological value
Rocks record strain, so who cares about stress?

These folds are beautiful examples of rock deformation (strain), but can they tell us anything about how they formed?
Rocks record strain, so who cares about stress?

- Forces applied to geologic materials produce deformation (strain)

- Plate tectonics is driven by gravitational body forces (which can be affected by temperature)

- So, if we want to understand what drives geodynamic behavior, we need to understand stress
Forces

- **Force:** A push or pull applied to a body. Force = mass x acceleration (Newton’s second law)
- **Units:** Newtons [N]; 1 N = 1 kg m s\(^{-2}\)
Forces

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- **Representation**: Vector

- **Example**: Gravity (directed toward center of Earth)

![Diagram of force F in the x and y directions](image-url)
Forces

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![Diagram of forces and component forces](image)
**Body forces versus surface forces**

- **Body force**: Forces that act throughout the volume of a solid. Proportional to its volume or mass.

  - Example: Slab pull (gravity)

- **Surface force**: Forces that act on the surface area bounding an element or volume. Proportional to the area upon which the force acts.

  - Example: Friction along a fault plane
Body and surface forces

- Consider a column of rock in the crust and the force acting on its basal surface \( (\sigma_{yy} \delta A) \)
- It is subject to gravity, so the base must support the weight of the overlying rock column

Fig. 2.1, Turcotte and Schubert, 2014
Body and surface forces

- Consider a column of rock in the crust and the force acting on its basal surface ($\sigma_{yy} \partial A$)

- The rock column has a thickness, $y$, and cross-sectional area, $\partial A$

- Thus, the weight is simply $\rho g y \partial A$, the density times gravitational acceleration times the volume of the column
Body and surface forces

- Consider a column of rock in the crust and the force acting on its basal surface ($\sigma_{yy} \partial A$).
- The rock column has a thickness, $y$, and cross-sectional area, $\partial A$.
- Thus, the weight is simply $\rho g y \delta A$, the density times gravitational acceleration time the volume of the column.
- Since the force at on the basal surface must equal the column weight, $\sigma_{yy} \partial A = \rho g y \partial A$, or $\sigma_{yy} = \rho g y$.
- Note, $\sigma_{yy}$ is acting vertically upward, balancing the downward weight of the column.

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Body and surface forces

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Surface stresses

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- **Surface stress**: A pair of equal and opposite forces acting on the area of a surface in a specific orientation
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**Surface stress:** A pair of equal and opposite forces acting on the area of a surface in a specific orientation

**Units:** Pascals [Pa]; $1 \text{ Pa} = 1 \text{ N m}^{-2}$

**Representation:** Pair of vectors with a specified surface area/orientation

**Example:** Hand pushing on table, table pushing back

![Surface stress diagram](image)

*after Twiss and Moores, 2006*
**Surface stresses**

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Surface stress

Component surface stresses

\[ \sigma_n \quad \sigma_s \]

\[ \sigma \quad \sigma_n \quad -\sigma_s \quad -\sigma \]

area \(A\)

after Twiss and Moores, 2006
Surface stress at the Moho

Assuming that the continental crust is 35 km thick and has an average density of 2750 kg m\(^{-3}\), how large is the vertical surface stress at its base?

You may want to report your value in megapascals (MPa, 1 million Pa).

\[ \sigma_{yy} = \text{???} \]

Fig. 2.1, Turcotte and Schubert, 2014
Crustal materials at mantle conditions

- Taken to high pressures and temperatures, crustal materials undergo transitions from one crystal form to another.

- What happens to these minerals when they undergo phase transitions?

- With this in mind, do you see any problems with the plot?
Continental buoyancy

- Consider a block of continental crust “floating” on the mantle
- The crust “floats” because a typical crustal density $\rho_c$ is $\sim 2750$ kg m$^{-3}$ whereas the mantle density $\rho_m$ is $\sim 3300$ km m$^{-3}$ (i.e., its buoyant)
- In hydrostatic equilibrium the weight of a crustal column of thickness $h$ is equal to that of the mantle beside it at the depth of its base $b$, or

$$\rho_c h = \rho_m b$$
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\[
\rho_c h = \rho_m b
\]

How high does a 35-km-thick continent stick out above the mantle?
Continental buoyancy

Continents are mostly near sea level

Oceans are mostly 4 km below
Horizontal surface stresses

- The horizontal surface stresses $\sigma_{xx}$ and $\sigma_{zz}$ act normal (or perpendicular) to vertical planes in the Earth.

- Combined $\sigma_{xx}$, $\sigma_{yy}$ and $\sigma_{zz}$ are the three normal stresses.

- In hot/weak rocks all three may be equal to the weight of the overlying rock or lithostatic pressure $p_L$

$$p_L \equiv \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \rho g y$$

- Note: We will use the orientation of the coordinate axes shown on the left in subsequent lectures.

Fig. 2.7, Turcotte and Schubert, 2014
Tangential surface stresses

- Forces and stresses can act both normal and parallel (or tangent) to surfaces, such as in strike-slip faults.
- These stresses, such as $\sigma_{xz}$ are called shear stresses.
- In the convention for labeling these stresses, the first letter is the direction normal to the surface element and the second letter is the direction of the shear force.

Fig. 2.10, Turcotte and Schubert, 2014
Tangential surface stresses

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- These stresses, such as $\sigma_{xz}$ are called shear stresses.
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- What is the orientation of $\sigma_{yx}$?
Stress in two dimensions

In two dimensions, we consider forces acting on four faces of an infinitesimal cube of dimension $\delta x \times \delta y \times \delta z$.

Here we assume no forces act or vary in the $z$ direction.

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Fig. 2.13, Turcotte and Schubert, 2014
Stress in two dimensions

Surface forces in 2D

- In two dimensions, we consider forces acting on four faces of an infinitesimal cube of dimension $\delta x \times \delta y \times \delta z$

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- Normal stresses: $\sigma_{xx}, \sigma_{yy}$

- Shear stresses: $\sigma_{xy}, \sigma_{yx}$

- At equilibrium we can state $\sigma_{xy} = \sigma_{yx}$

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- Why?
Changing coordinate reference frames

- The normal and shear stresses in the coordinate system $x, y$ can also be represented in another coordinate system $x', y'$ that has been rotated by angle $\theta$.

- Knowing $\sigma_{xx}, \sigma_{yy}$ and $\sigma_{xy}$, their equivalent values in the rotated coordinate system $x', y'$ are

  \[
  \sigma_{x'x'} = \sigma_{xx} \cos^2 \theta + \sigma_{yy} \sin^2 \theta + \sigma_{xy} \sin 2\theta
  \]
  \[
  \sigma_{y'y'} = \sigma_{xx} \sin^2 \theta + \sigma_{yy} \cos^2 \theta - \sigma_{xy} \sin 2\theta
  \]
  \[
  \sigma_{x'y'} = \frac{1}{2} (\sigma_{yy} - \sigma_{xx}) \sin 2\theta + \sigma_{xy} \cos 2\theta
  \]
Changing coordinate reference frames

- Why would we want to do this?

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As it turns out, for any state of stress \( \sigma_{xx}, \sigma_{yy}, \text{and} \sigma_{xy} \), it is possible to find a stress state with no shear stresses.

This can be found by setting \( \sigma_{x'y'} \) equal to zero and solving for \( \theta \) in the equation below, which yields:

\[
\tan 2 \theta = \frac{2 \sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}
\]

The orientation of \( \theta \) is the principal axis of stress, as is \( \theta + \pi/2 \).
Changing coordinate reference frames

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• Why would we want to do this?

• As it turns out, for any state of stress $\sigma_{xx}$, $\sigma_{yy}$ and $\sigma_{xy}$ it is possible to find a stress state with no shear stresses.

• This can be found by setting $\sigma_{x'y'}$ equal to zero and solving for $\theta$ in the equation below:

$$\sigma_{x'y'} = \frac{1}{2} (\sigma_{yy} - \sigma_{xx}) \sin 2\theta + \sigma_{xy} \cos 2\theta$$

which yields

$$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

• The orientation of $\theta$ is the principal axis of stress, as is $\theta + \pi/2$.
Principal stresses

- The normal stresses in the principal axis coordinate system are known as the **principal stresses** $\sigma_1$ and $\sigma_2$

- The principal stresses can be found from stresses in coordinate system $x, y$ using

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \left[ \frac{(\sigma_{xx} - \sigma_{yy})^2}{4} + \sigma_{xy}^2 \right]^{1/2}$$

- Conversely, we can get the stresses in coordinate system $x, y$ from the principal stresses using

$$\sigma_{xx} = \frac{\sigma_1 + \sigma_2}{2} + \frac{(\sigma_1 - \sigma_2)}{2} \cos 2\theta$$

$$\sigma_{xy} = -\frac{1}{2}(\sigma_1 - \sigma_2) \sin 2\theta$$

$$\sigma_{yy} = \frac{\sigma_1 + \sigma_2}{2} - \frac{(\sigma_1 - \sigma_2)}{2} \cos 2\theta$$
Stress in three dimensions

- In three dimensions, we consider forces acting on all six faces of an infinitesimal cube of dimension $\delta x \times \delta y \times \delta z$
- Normal stresses: $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$
- Shear stresses: $\sigma_{xy}, \sigma_{yx}, \sigma_{xz}, \sigma_{zx}, \sigma_{yz}, \sigma_{zy}$
- At equilibrium we can state $\sigma_{xy} = \sigma_{yx}, \sigma_{xz} = \sigma_{zx}, \sigma_{yz} = \sigma_{zy}$
Stress in three dimensions

- As before, we can also determine the principal stresses in three dimensions
- The convention is that $\sigma_1 \geq \sigma_2 \geq \sigma_3$
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- The convention is that $\sigma_1 \geq \sigma_2 \geq \sigma_3$.
- **Isotropic stress** is when $\sigma_1 = \sigma_2 = \sigma_3 = p$. 
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- Isotropic stress is when $\sigma_1 = \sigma_2 = \sigma_3 = p$.
- The hydrostatic state of stress is when the normal stresses equal $p$ and there are no shear stresses.
Stress in three dimensions

- As before, we can also determine the principal stresses in three dimensions.
- The convention is that $\sigma_1 \geq \sigma_2 \geq \sigma_3$.
- Isotropic stress is when $\sigma_1 = \sigma_2 = \sigma_3 = p$.
- The hydrostatic state of stress is when the normal stresses equal $p$ and there are no shear stresses.
- The lithostatic stress is the hydrostatic state of stress where stress increases in proportion to the density of the overlying rock.
Stress in three dimensions

- When the principal stresses are not equal

\[
p = \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3} (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})
\]
Deviatoric stresses

- We often subtract pressure from normal stresses to determine the deviatoric stresses (indicated by primes)

\[
\begin{align*}
\sigma'_{xx} &= \sigma_{xx} - p \\
\sigma'_{xy} &= \sigma_{xy} \\
\sigma'_{xx} &= \sigma_{yy} - p \\
\sigma'_{xz} &= \sigma_{xz} \\
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- The same can be done for the principal stresses

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• What is the sum of the deviatoric principal stresses?

• For rock deforming by viscous flow, all deformation is the result of a nonzero deviatoric stress
Recap

- A surface stress is a pair of equal and opposite forces applied to an area.

- Stating the stress in two-dimensions requires knowing three stress values (2 normal, 1 shear); in 3D, we must know six stress values (3 normal, 3 shear).

- Deformation of the lithosphere occurs as the result of applied stresses and understanding these stresses is integral to understanding geodynamic processes in the Earth.
References
